May 4th, 2023 **Name (Please Print)** ______________________

TAS/TDS- Final Exam - **Maximum Time: 3 hours** **Maximum Score: 100**

1. (15 points) Let $(\Gamma = (V, E), \mu)$ be a weighted graph. For any $A \subset V$, define the potential operator $G_A : C(V) \to \mathbb{R}$ by
$$G_A f(x) = \sum_{y \in V} g_A(x, y) f(y) \mu_y.$$
Find $\Delta_A G_A$ when $A \subset V$, $A \neq V$ and Γ is recurrent. Show that
$$\Delta G f = -f$$
when Γ is transient and $f \in L^1(V)$.

2. (15 points) Let S_n be the simple symmetric walk on $\mathbb{Z}^d$. Let
$$\tau_R = \inf\{n \geq 0 :\mid S_n \mid = R\}.$$
Let $h : \mathbb{Z}^d \to [0, \infty)$ be given by
$$h(x) = \mathbb{P}_x(\tau_{20} < \tau_1).$$
Show that
   (a) $h(x) = 1$ whenever $\mid x \mid \geq 20$
   (b) $h(x) = 0$ whenever $\mid x \mid \leq 1$
   (c) h is harmonic on the set $1 <\mid x \mid< 20$, i.e.
$$h(x) = \frac{1}{2d}\left(\sum_{i=1}^{d} h(x + e_d) + h(x - e_d)\right),$$
   whenever $1 <\mid x \mid< 20$, where $\{e_i : 1 \leq i \leq d\}$ are the standard basis for $\mathbb{Z}^d$.

3. (20 points) Assume the following version of:
Cramer's Theorem: Let (X_i) be i.i.d. $\mathbb{R}$–valued random variables such that
$$0 \in \text{interior}\{t \in \mathbb{R} : \varphi(t) = \mathbb{E}\, e^{tX_1} < \infty\} \tag{1}$$
Let $S_n = \sum_{i=1}^{n} X_i$. Then for all $a > \mathbb{E}X_1$
$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(S_n \geq an) = -I(a), \tag{2}$$
where
$$I(z) = \sup_{t\in\mathbb{R}}[zt - \log \varphi(t)].$$
Find I: when $X_i \sim$
   (a) Exponential (α), for some $\alpha > 0$
   (b) Normal(α,1), for some $\alpha > 0$

4. (15 points) We place M marbles in P pots. At each time unit we choose one of the marbles uniformly at random and place it in one of the urns also uniformly chosen at random. Denote by M_n to be the number of marbles in the first urn at time n. Find a_n, b_n, so that $a_n M_n + b_n$ is a martingale.

5. (15 points) Consider a martingale where Z_n can take on only the values 2^{-n-1} and $1 - 2^{-n-1}$, each with probability $\frac{1}{2}$.

(a) Given that Z_n, conditional on Z_{n-1}, is independent of $Z_{n-2}, Z_{n-3}, \dots, Z_1$ find $E[Z_n \mid Z_{n-1}]$ for each n so that the martingale condition is satisfied.

(b) Show that $\mathbb{P}(\sup_{n\geq 1} Z_n \geq 1) = \frac{1}{2} \neq 0 = \mathbb{P}(\bigcup_{n\geq 1}\{Z_n \geq 1\})$

(c) Show that for all $\epsilon > 0$, $\mathbb{P}(\sup_{n\geq 1} Z_n \geq a) \leq \frac{\mathbb{E}[Z_1]}{a-\epsilon}$.

6. (20 points) Let (Γ, μ) be a weighted graph that is connected and locally finite. Let X_n be a random walk on (Γ, μ) and $p_n(\cdot,\cdot)$ be the n-th step transition density for $n \geq 1$ with P being the one step transition operator.

(a) State the Nash inequality N_α for Γ

(b) Fix $x \in V$ and $n \geq 1$. Let $r_n^x(\cdot) = p_n(x,\cdot) + p_{n+1}(x,\cdot)$ and $\phi_n(x) = r_{2n}^x(x)$. Assume Γ satisfies (N_α) for $\alpha \geq 1$

i. Show that

$$\phi_{n+1}(x) - \phi_n(x) = -\mathcal{E}(r_n^x, r_n^x) \leq -2^{-\frac{4}{\alpha}} C_N \phi_n(x)^{1+\frac{2}{\alpha}} \tag{3}$$

ii. Conclude that there exists $c_1 > 0$ such that

$$\phi_{n+1}(x) - \phi_n(x) \leq c_1 \phi_n(x)^{1+\frac{2}{\alpha}}$$

and that this implies that there exists $c_2 > 0$ such that

$$\phi_n(x) \leq c_2^{-\frac{\alpha}{2}} n^{-\frac{\alpha}{2}} \tag{4}$$

(c) Show that if Γ satisfies (N_α) for $\alpha \geq 1$ then there exists $c_3 > 0$ such that

$$p_n(x,x) \leq \frac{c_3}{\max(1,n)^{\frac{\alpha}{2}}}$$

for all $x \in V$ and $n \geq 0$.